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On Differential Relations with Lower Continuous Right-Hand Side. An Existence Theorem*

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1. INTRODUCTION

In this paper we consider differential relations of the form

$$x' \in F(t, x), \tag{1.1}$$

where F is a multifunction that is defined on a suitable subset $I \times B$ of $\mathbb{R}^{n+1}$ and takes values in the family of nonempty, compact but not necessarily convex, subsets of $\mathbb{R}^n$. Assuming that F is continuous with respect to x, the existence of solutions for (1.1) has been proved recently by several different techniques [1–6]. In particular, in [1] a selection theorem is first established, which yields the existence of a solution of (1.1) as a direct consequence of Schauder's fixed-point theorem.

Our present aim is to prove an existence theorem for lower continuous differential relations. Lower continuous orientor fields often arise in control theory. Namely, if

$$x' = f(t, x, u), \qquad u \in \mathscr{U}, \tag{1.2}$$

describes a control system, we may ask about solutions $t \to x(t, u(t))$ of (1.2) for which $x'(t, u)$ belongs a.e. to the closure $F^*(t, x)$ of the set of extreme points of

$$F(t, x) = \{f(t, x, u): u \in \mathscr{U}\}.$$

Clearly, it F is Hausdorff continuous and compact valued, F^* is lower continuous. In the linear case, this subject had a deep investigation leading to the bang–bang principle. In the general case nothing can be said unless some solutions for

$$x' \in F^*(t, x)$$

are provided. Now let $\mu_{t,x}(\cdot)$ denote a measure, for instance, a probability distribution on the values $u'(t, x)$ of derivatives u' at (t, x). If $\mu_{t,x}$ depends on

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(t, x) in a weakly continuous way, and some fixed compact X contains the support of $\mu_{t,x}$ for all t and x, then the mapping

$$(t, x) \to \mathrm{Supp}(\mu_{t,x})$$

affords another example of a lower continuous field.

The proof of our existence theorem is similiar in the main lines to the one written in [1] for the continuous case. For each absolutely continuous $u: I \to \mathbb{R}$ we consider the set $G(u)$ of the integrable mappings v such that $v(t) \in F(t, u(t))$ at almost each $t \in I$. The mapping $u \to G(u)$ is then lower continuous. Our main theorem states that G admits a continuous selection. If G takes only convex values, this theorem becomes a consequence of a well-known result of Michael [8]. In the general case $G(u)$ is not convex; however, we can still define some kind of combinations of arbitrary elements $v_1, \ldots, v_m$ of $G(u)$: the mappings v such that, for some measurable partition $(J_1, \ldots, J_m)$ of I, $v(t) = v_i(t)$ iff $t \in J_i$, $(i = 1, \ldots, m)$. Clearly $v \in G(u)$. This property of G, together with accurate choices of the aforementioned partitions, based upon a systematic use of continuous partitions of unity, enables us to construct a suitable sequence of approximate selections. The limit of this sequence is shown to exist and to be a continuous selection. This fact allows us to deduce the existence of solutions for (1.1) by means of Schauder's theorem.

2. Notations and Basic Definitions

Throughout this paper we consider a mapping F defined in a compact subset R of $\mathbb{R}^{n+1}$, with values in the family comp X of nonempty compact subsets of a bounded subspace X of $\mathbb{R}^n$. For simplicity, we assume that $X \subseteq \mathbb{R}^n$ is a closed ball with center at the origin and radius $M > 0$, and that $R = I \times B$, where $B \subseteq \mathbb{R}^n$ is a closed ball with center at the origin and radius $b > 0$, and where I is the nondegenerate closed interval $[0, T]$, with $T \leqslant b/M$.

We use the symbol $d(\cdot, \cdot)$ for the metric in point-spaces, such as R and X and use $D(\cdot, \cdot)$ for the Hausdorff distance in comp X. We write $d(x, A)$ for the distance $\inf\{d(x, a); a \in A\}$ from the point x to the set A, $\Delta(A)$ for the diameter of the set A, and $B[A, \epsilon]$ for the closed ball $\{b: d(b, A) \leqslant \epsilon\}$ of radius $\epsilon > 0$ about the set A. $\mu(\cdot)$ denotes the Lebesgue measure on I. Recall that a mapping $F: R \to \mathrm{comp}\, X$ is lower continuous iff

$$\forall z_0 \in R, \quad \forall \epsilon > 0, \quad \exists \delta > 0: d(z, z_0) \leqslant \delta \Rightarrow F(z_0) \subseteq B[F(z), \epsilon]. \tag{2.1}$$

Let $\mathscr{C}$ and $\mathscr{L}^1$ denote the Banach spaces of continuous and Lebesgue integrable mappings of I into $\mathbb{R}^n$, respectively, and define $\mathscr{K}$ as the family of absolutely continuous mappings $u: I \to \mathbb{R}^n$ such that $u(0) = 0$ and $\| u'(t)\| \leqslant M$ at almost every $t \in I$. $\mathscr{K}$ is known to be a nonempty, compact and convex subset of $\mathscr{C}$. Finally, for any $u \in \mathscr{K}$, we define $G(u)$ by

$$G(u) = \{v \in \mathscr{L}^1 \colon v(t) \in F(t, u(t)) \text{ a.e. in } I\}. \tag{2.2}$$

Some results on $G(u)$ will be given in the next section.

3. Preliminary Results

Proposition 1. *Let F be lower continuous, $u_0 \in K$, $\epsilon > 0$ and $\epsilon' > 0$. Then there exists a $\rho > 0$ and a compact $E \subseteq I$ such that*

$$\mu(I - E) \leqslant \epsilon', \tag{3.1}$$

$$\| u - u_0 \| \leqslant \rho \Rightarrow F(t, u_0(t)) \subseteq B[F(t, u(t)), \epsilon], \qquad \forall t \in E, \quad \forall u \in \mathscr{K}. \tag{3.2}$$

Proof. The mapping $t \to F(t, u_0(t)$ is lower continuous, hence measurable. According to [9], a compact $E \subseteq I$ can be found, with $\mu(I - E) \leqslant \epsilon'$, such that the restriction of the map $t \to F(t, u_0(t))$ to E is continuous. Hence for some $\sigma > 0$

$$| t_1 - t_2 | \leqslant \sigma \Rightarrow D(F(t_1, u_0(t_1)), F(t_2, u_0(t_2))) \leqslant \epsilon/2, \qquad \forall t_1, t_2 \in E, \tag{3.3}$$

and we can cover the compact set $E_0 = \{(t, u_0(t)) \colon t \in E\} \subseteq R$ with a finite number of open balls $(B_i)_{1 \leqslant i \leqslant m}$ with centers at the points $c_i = (t_i, u_0(t_i))$ and radii ρ_i, where, for $1 \leqslant i \leqslant m$, $t_i \in E$, $\rho_i \leqslant \sigma$, and

$$d((t, x), (t_i, u_0(t_i))) \leqslant \rho_i \Rightarrow F(t_i, u_0(t_i)) \subseteq B[F(t, x), \epsilon/2], \qquad \forall (t, x) \in R. \tag{3.4}$$

Clearly, if $t \in E$ and $(t, x) \in B_i$ for some i, then

$$F(t, u_0(t)) \subseteq B[F(t_i, u_0(t_i)), \epsilon/2] \subseteq B[F(t, x), \epsilon]. \tag{3.5}$$

We can now choose a $\rho > 0$ for which

$$B[E_0, \rho] \subseteq \bigcup_{i=1}^{m} B_i, \tag{3.6}$$

so that condition (3.2) holds.

Proposition 2. *Let F be lower continuous. Then, for any $u \in \mathscr{K}$, $G(u)$ is a nonempty closed, and bounded subset of $\mathscr{L}^1$. Moreover, the mapping $u \to G(u)$ is lower continuous.*

Proof. Being measurable, the mapping $t \to F(t, u(t))$ admits some measurable selection f (see [7]), which is an element of $G(u)$. Hence $G(u) \neq \varnothing$. Let now

$(f_n)_{n \geqslant 1}$ be a sequence of functions in $G(u)$ that converges to $\bar{f}$ in the $\mathscr{L}^1$-norm. This implies that, for a suitable subsequence (f_{n_k}),

$$\lim_{k \to \infty} f_{n_k}(t) = \bar{f}(t) \qquad \text{a.e. in } I,$$

so that $\bar{f}(t) \in F(t, u(t))$ a.e., because $F(t, u(t))$ is closed. Thus $\bar{f} \in G(u)$. The boundedness of $G(u)$ is obvious.

Finally, let $u_0 \in \mathscr{K}$ and $\epsilon > 0$ be given. According to Proposition 1, we can choose a $\rho > 0$ and a compact set $E \subseteq I$ that satisfy

$$\begin{gathered} \mu(I - E) \leqslant \epsilon/4M, \\ \| u - u_0 \| \leqslant \rho \Rightarrow F(t, u_0(t)) \subseteq B[F(t, u(t)), \epsilon/2T], \qquad \forall t \in E, \quad \forall u \in \mathscr{K}. \end{gathered} \tag{3.8}$$

Take now any $f_0 \in G(u_0)$ and any $u \in \mathscr{K}$, with $\| u - u_0 \| \leqslant \rho$. Our last thesis will be proved if we exhibit an $f \in G(u)$ such that $\| f - f_0 \| \leqslant \epsilon$. To do this, we define a measurable mapping $H: I \to \operatorname{comp} X$ as follows:

$$\begin{aligned} H(t) &= F(t, u(t)) \cap B[\{f_0(t)\}, \epsilon/2T] && \text{if} \quad t \in E, \\ &= F(t, u(t)) && \text{if} \quad t \notin E, \end{aligned} \tag{3.9}$$

and we select a measurable $f: I \to X$ such that $f(t) \in H(t)$ for all $t \in I$. Clearly

$$\begin{aligned} \| f - f_0 \|_1 &= \int_E \| f(t) - f_0(t) \| \, dt + \int_{I-E} \| f(t) - f_0(t) \| \, dt \\ &\leqslant T \cdot \epsilon/2T + 2M \cdot \epsilon/2M = \epsilon. \end{aligned}$$

Q.E.D.

4. A Selection Theorem

THEOREM 1. *Let $F: R \to \operatorname{comp} X$ be lower continuous. Then there exists a continuous mapping $g: \mathscr{K} \to \mathscr{L}^1$ such that, for every $u \in \mathscr{K}$, we have $g(u)(t) \in F(t, u(t))$ at almost every $t \in I$.*

Otherwise stated, if F is lower continuous, then the lower continuous set-valued mapping G defined by (2.2) admits a continuous selection.

We begin our proof by constructing an approximating sequence of mappings $(g_n)_{n \geqslant 0}$ from $\mathscr{K}$ to $\mathscr{L}^1$, in such a way that the following recursive conditions are satisfied:

$(\mathrm{a})_n$ $\quad g_n$ is continuous,

$(\mathrm{b})_n$ $\quad \forall u \in \mathscr{K}$, $\mu(W_n(u)) \leqslant 2^{-n}$, where

$$W_n(u) = \{t \in I: d\,(g_n(u)(t), F(t, u(t))) > M \cdot 2^{-n}\}, \tag{4.1}$$

(c)$_n$ $\forall u \in \mathscr{K}$, $\mu(\{t \in I: d(g_n(u)(t), g_{n-1}(u)(t)) > M \cdot 2^{-n+1}\}) \leqslant 2^{-n+3}$,

(d)$_n$ $\exists \delta_n > 0$ such that, if $V \subseteq \mathscr{K}$ and $\Delta(V) \leqslant \delta_n$, then

$$\mu\left(\bigcup_{u \in V} W_n(u)\right) \leqslant 2^{-n+1}, \tag{4.2}$$

and

$$\mu(\{t \in I: \exists u, v \in V; g_n(u)(t) \neq g_n(v)(t)\}) \leqslant 2^{-n+3}. \tag{4.3}$$

Define $g_0(u)(t) = 0$ for every $u \in \mathscr{K}$ and every $t \in I$. Conditions (a)$_0$ to (d)$_0$ are then trivially fulfilled. Let now g_{n-1} be constructed. Choose δ_{n-1} in such a way that the condition in (d)$_{n-1}$ is satisfied. For each $u \in \mathscr{K}$, by Proposition 1, we have a $\rho = \rho(u) > 0$ and a compact set $E = E(u) \subseteq I$ for which

$$0 < \rho(u) < \delta_{n-1}/2, \tag{4.4}$$

$$\mu(I - E(u)) \leqslant 2^{-n}, \tag{4.5}$$

$$d(x, u(t)) \leqslant \rho(u) \Rightarrow F(t, u(t)) \subseteq B[F(t, x), M \cdot 2^{-n}], \qquad \forall t \in E(u). \tag{4.6}$$

Since $\mathscr{K}$ is compact, it can be covered with a finite number of open balls $(U_i)_{1 \leqslant i \leqslant m}$ with centers at some points u_i and radii $\rho_i = \rho(u_i)$. Construct now a continuous partition of unity $(p_i)_{1 \leqslant i \leqslant m}$ subordinate to the covering (U_i). For each i, choose $v_i \in G(u_i)$ such that

$$d(v_i(t), g_{n-1}(u_i)(t)) \leqslant 2^{-n+1} \cdot M, \qquad \forall t \notin W_{n-1}(u_i). \tag{4.7}$$

The proof of Proposition 2 shows that this is possible. In analogy with [1], we shall define g_n letting $g_n(u)(t)$ coincide with $v_i(t)$ whenever $t \in J_i(u)$, where the sets $(J_i(u))_{1 \leqslant i \leqslant m}$ form a partition of I and have measures proportional to the coefficients $p_i(u)$. The fulfillment of (a)$_n$ to (d)$_n$ requires a very careful construction of the sets $J_i(u)$.

Recalling (4.6), for $1 \leqslant i \leqslant m$ we define

$$A_i^+ = E(u_i), \qquad A_i^- = I \backslash A_i^+. \tag{4.8}$$

By (4.5) we have

$$\mu(A_i^+) \geqslant T - 2^{-n}, \qquad \mu(A_i^-) \leqslant 2^{-n}. \tag{4.9}$$

Let Γ^m be the family of m-vectors $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that

$$\sum_{i=1}^{m} \lambda_i = T, \qquad 0 \leqslant \lambda_i \leqslant T \qquad (i = 1, \ldots, m).$$

For $\lambda \in \Gamma^m$, we construct the partition $(J_1(\lambda), \ldots, J_n(\lambda))$ inductively as follows: Let $J_k(\lambda)$ be defined for any $k < i$. Then we set

$$\begin{aligned} Y_i(\lambda) &= I \Big\backslash \bigcup_{k<i} J_k(\lambda), \\ Y_i^+(\lambda) &= Y_i(\lambda) \cap A_i^+, \qquad Y_i^-(\lambda) = Y_i(\lambda) \cap A_i^-. \end{aligned} \tag{4.10}$$

We define a mapping $\varphi_{i,\lambda}$ from $Y_i(\lambda)$ to $\mathbb{R}$ by the condition

$$\begin{aligned} \varphi_{i,\lambda}(t) &= \mu([0, t] \cap Y_i^+(\lambda)) && \text{if} \quad t \in Y_i^+(\lambda), \\ &= \mu([0, t] \cap Y_i^-(\lambda)) + \mu(y_i^+(\lambda)) && \text{if} \quad t \in Y_i^-(\lambda), \end{aligned} \tag{4.11}$$

and take $J_i(\lambda)$ to be the inverse image of the interval $[0, \lambda_i]$:

$$J_i(\lambda) = \varphi_{i,\lambda}^{-1}([0, \lambda_i]). \tag{4.12}$$

For $u \in \mathscr{K}$ and $1 \leqslant i \leqslant m$ we now define $Y_i(u)$, $\varphi_{i,u})$ and $J_i(u)$ as given by (4.10), (4.11), and (4.12), respectively, for $\lambda = (T \cdot p_1(u), \ldots, T \cdot p_m(u))$. Finally we define

$$g_n(u)(t) = \sum_{i=1}^{m} \chi[J_i(u)](t) \cdot v_i(t), \tag{4.13}$$

where $\chi[J_i(u)]$ is the characteristic function of $J_i(u)$.

We claim that conditions $(a)_n$ to $(d)_n$ hold.

Let us start proving $(b)_n$. If $u \in \mathscr{K}$ and $t \in J_i(u) \cap A_i^+$ for some i, then (4.6) shows that $t \notin W_n(u)$. Hence

$$W_n(u) \subseteq \bigcup_{1 \leqslant i \leqslant m} (J_i(u) \cap A_i^-). \tag{4.14}$$

We shall deduce $(b)_n$ by showing that

$$\mu\left(\bigcup_{1 \leqslant i \leqslant m} (J_i(u) \cap A_i^+)\right) \geqslant T - 2^{-n}. \tag{4.15}$$

The $\varphi_{i,u}$ are measure preserving mappings; thus

$$\mu(J_i(u)) = T \cdot p_i(u) \qquad (1 \leqslant i \leqslant m). \tag{4.16}$$

By construction, we also have

$$\mu(J_i(u) \cap A_i^+) \geqslant \inf\{T \cdot p_i(u), \mu(Y_i^+(u)\} \geqslant \inf\{T \cdot p_i(u), \mu(Y_i(u)) - 2^{-n}\}. \tag{4.17}$$

Let ν be the smallest integer for which

$$\sum_{i \leqslant \nu} p_i(u) > 1 - 2^{-n}/T. \tag{4.18}$$

If $i < \nu$, by (4.18), $T \cdot p_i(u) \leqslant \mu(Y_i(u) - 2^{-n}$; thus (4.17) implies

$$\mu(J_i(u) \cap A_i^+) = T \cdot p_i(u). \tag{4.19}$$

Furthermore, by (4.18), $T \cdot p_\nu(u) > T - 2^{-n} - \sum_{i<\nu} T \cdot p_i(u)$, and from (4.17) we infer

$$\mu(J_\nu(u) \cap A_\nu^+) \geqslant T - 2^{-n} - \sum_{i<\nu} T \cdot p_i(u). \tag{4.20}$$

Summing (4.19) with (4.20), we obtain

$$\sum_{i=1}^{m} \mu(J_i(u) \cap A_i^+) \geqslant \sum_{i=1}^{\nu} \mu(J_i(u) \cap A_i^+) \geqslant T - 2^{-n}, \tag{4.21}$$

which proves (4.15).

To deduce $(\mathrm{c})_n$, for $u \in \mathscr{K}$, call $\mathscr{T}(u)$ the set of indices i such that $p_i(u)$ is strictly positive. Because of (4.4), $\| u_i - u \| \leqslant \delta_{n-1}/2$ for any $i \in \mathscr{T}(u)$ hence the set $\{u, u_i : i \in \mathscr{T}(u)\}$ has a diameter smaller than δ_{n-1}, and $(\mathrm{d})_{n-1}$ provides the existence of two sets $\mathscr{N}_1, \mathscr{N}_2 \subseteq I$ such that

$$\mu(\mathscr{N}_1) \leqslant 2^{-n+2}, \qquad \mu(\mathscr{N}_2) \leqslant 2^{-n+1}, \tag{4.22}$$

$$\mathscr{N}_1 \supseteq \left(\bigcup_{i \in \mathscr{T}(u)} W_{n-1}(u_i) \right), \tag{4.23}$$

and

$$g_{n-1}(u_i)(t) = g_{n-1}(u)(t) \qquad \text{for any} \qquad i \in \mathscr{T}(u), \quad t \notin \mathscr{N}_2 . \tag{4.24}$$

Let $t \notin \mathscr{N}_1 \cup \mathscr{N}_2$. Then $g_n(u)(t) = v_i(t)$ for some $i \in \mathscr{T}(u)$; furthermore, by (4.7) and (4.23), $d(v_i(t), g_{n-1}(u_i)(t)) \leqslant M \cdot 2^{-n+1}$, and by (4.24), $g_{n-1}(u_i)(t) = g_{n-1}(u)(t)$. Hence $d(g_n(u)(t), g_{n-1}(u)(t)) \leqslant M \cdot 2^{-n+1}$, which yields $(\mathrm{c})_n$, because $\mu(\mathscr{N}_1 \cup \mathscr{N}_2) \leqslant 2^{-n+3}$.

We come now to the proof of $(\mathrm{d})_n$. For $V \subseteq \Gamma^m$ and $\Delta(V) \leqslant \delta$, we consider the set $\Lambda \subseteq I$ of the points assigned to different J_i's by two suitable elements of V:

$$\Lambda = \{t \in I : \exists i, \exists \lambda, \lambda' \in V; t \in J_i(\lambda), t \notin J_i(\lambda')\}. \tag{4.25}$$

By setting

$$\Lambda_k = \{t \in I : \exists \lambda, \lambda' \in V; t \in J_k(\lambda), t \notin J_k(\lambda')\}, \tag{4.26}$$

so that $\Lambda = \bigcup_k \Lambda_k$, it can be seen by induction that $\mu(\Lambda_k) \leqslant k \cdot \delta$, hence

$$\mu(\Lambda) \leqslant m^2 \cdot \delta. \tag{4.27}$$

From (4.27) and the equicontinuity of the family of mappings $(p_i)_{1 \leqslant i \leqslant m}$ we infer the existence of a $\delta_n > 0$ such that $V \subseteq \mathscr{K}$ and $\Delta(V) \leqslant \delta_n$ imply

$$\mu(\{t \in I : \exists_i, \exists u, v \in V; t \in J_i(u), t \notin J_i(v)\}) \leqslant 2^{-n}, \tag{4.28}$$

which immediately yields (4.3). Furthermore, using (4.14) and (4.28) we obtain

$$\mu\left(\bigcup_{\mu \in V} W_n(u)\right) \leqslant \mu\left(\bigcup_{\mu \in V}\left(\bigcup_{i=1}^{m} (J_i(u) \cap A_i^-)\right)\right) \leqslant 2^{-n+1}. \tag{4.29}$$

Finally, $(a)_n$ is seen to be a consequence of (4.27) and of the equicontinuity of the mappings $(p_i)_{1 \leqslant i \leqslant m}$.

We have thus proved the existence of an approximating sequence (g_n) with the desired properties. For any $u \in \mathscr{K}$, $(b)_n$, and $(c)_n$ show that

$$d(g_n(u), G(u)) \leqslant 2^{-n} \cdot T \cdot M + 2^{-n} \cdot 2M, \tag{4.30}$$

$$d(g_n(u), g_{n-1}(u)) \leqslant 2^{-n+1} \cdot T \cdot M + 2^{-n+3} \cdot 2M, \tag{4.31}$$

where $d(\cdot, \cdot)$ is the distance induced by the norm in $\mathscr{L}^1$. By (4.31), for each $u \in \mathscr{K}$ $(g_n(u))_{n \geqslant 0}$ is a Cauchy sequence. Then the completeness of $\mathscr{L}^1$ provides the existence of a limit $g(u) \in \mathscr{L}^1$. The mapping $u \to g(u)$, being a uniform limit of continuous mappings, is also continuous. Moreover, by (4.30),

$$\lim_{n \to \infty} d(g_n(u), G(u)) = 0. \tag{4.32}$$

Thus $g(u) \in G(u)$, because $G(u)$ is closed, and our proof is complete.

Remark. The lower continuity of F from the product space $I \times B$ to comp X, assumed in the preceding proof, cannot be replaced by the lower continuity of F with respect to the variables t and x separately, because this assumption does not provide the measurability of the composite mappings $t \to F(t, u(t))$, for $u \in \mathscr{K}$.

Nevertheless, it can be seen that the hypothesis of Theorem 1 can be somewhat weakened; in fact the following can be proved:

Let $(I_k)_{k \geqslant 1}$ be a countable measurable covering of I. If the restriction of F to each product space $I_k \times B$ is lower continuous, then the conclusion of Theorem 1 still holds.

5. An Existence Theorem

THEOREM 2. *If F: $R \to$ comp X is lower continuous, there exists an absolutely continuous mapping $\hat{u}$ of I into $\mathbb{R}^n$ such that $\hat{u}(0) = 0$ and $\hat{u}'(t) \in F(t, \hat{u}(t))$ at almost every $t \in I$.*

Indeed, Theorem 1 enables us to construct a continuous mapping g: $\mathscr{K} \to \mathscr{L}^1$ such that, for each $u \in \mathscr{K}$, $g(u)(t) \in F(t, u(t))$ at almost every $t \in I$. For each $u \in \mathscr{K}$, let $h(u)$ be the mapping of I into $\mathbb{R}^n$ defined by

$$h(u)(t) = \int_0^t g(u)(s)\, ds. \tag{5.1}$$

Clearly, h is a continuous mapping of $\mathscr{K}$ into itself. Hence, by Schauder's theorem, there exists a point $\hat{u} \in \mathscr{K}$ for which $\hat{u} = h(\hat{u})$ i.e., $\hat{u}(t) = h(\hat{u})(t)$ at every $t \in I$. This implies $\hat{u}(0) = 0$ and $\hat{u}'(t) = g(\hat{u})(t) \in F(t, \hat{u}(t))$ at almost every $t \in I$.

COROLLARY. *Let F: $R \to$ comp X be continuous. Then there exists an absolutely continuous mapping $\hat{u}$: $I \to \mathbb{R}^n$ such that $\hat{u}(0) = 0$ and $\hat{u}'(t) \in F^*(t, \hat{u}(t))$ at almost every $t \in I$, where $F^*(t, x)$ denotes the closure of the set of extreme points of $F(t, x)$.*

Namely, the mapping $(t, x) \to F^*(t, x)$ is lower continuous.

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